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M – 1458

Reg. No. : ....

Name : ....

Fifth Semester B.Sc. Degree Examination, December 2021

First Degree Programme under CBCSS

Mathematics

Core Course VII

MM 1543 – ABSTRACT ALGEBRA – GROUP THEORY

(2018 & 2019 Admission)

Time : 3 Hours

Max. Marks : 80

SECTION – I

(**All** questions are compulsory. These questions carry **1** mark each)

1. Define an associative binary operation.

2. Let a and b belong to a group G. Find an x in G such that $xabx^{-1} = ba$.

3. Define the centre of a group.

4. Find μ^{100} if $\mu = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 3 & 1 & 6 \end{bmatrix}$.

5. Find the order of the permutation $(23)(156)$.

6. Find $Aut(z)$.

7. Define normal subgroup.

P.T.O.

8. What is the order of the factor group $\frac{z60}{<15>}$.

9. Find the Kernel of the mapping $\varphi : R^* \to R^*$ defined by $\varphi(x) = |x|$.

10. Find the left cosets of $H\{0,\ In,\ Iz, ...\}$ in Z where n is a positive integer.

SECTION – II

(Answer any **eight** questions. These questions carry **2** marks each.)

11. Prove that the left and right cancellation laws hold in a group.

12. Prove that a group G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all a and b in G.

13. Prove that for each a in a group G, the centralizer of a is a subgroup of G.

14. Find all generators of z_{10} and z_{12}.

15. Prove that every cyclic group is abelian.

16. Express $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 4 & 3 & 1 \end{bmatrix}$ as a product of cycles.

17. Prove that for $n > 1$, A_n has order $\frac{n!}{2}$.

18. Let $\varphi : G \to \overline{G}$ is an isomorphism. The prove that G is abelian if and only if $\overline{G}$ is abelian.

19. Show that z has infinitely many subgroups isomorphic to z.

20. Let H be a subgroup of G. Then prove that $aH = bH$ if and only if $a^{-1}b \in H$.

21. Let G be a group and $a \in G$. Show that $a^{|G|} = e$.

22. Let $|a| = 30$. How many left cosets of $\langle a^4 \rangle$ in $\langle a \rangle$ are there? List them.

23. Prove that the centre $Z(G)$ of a group G is normal.

24. Prove that a factor group of an abelian group is abelian.

25. Prove that a normal subgroup N is the Kernel of the mapping $g \to gN$ from G to G/N.

26. Prove that the mapping $\varphi : GL(Z,R) \mapsto R^*$ defined by $\varphi(A) = \det A$ is a homomorphism.

SECTION – III

(Answer any **six** questions. These questions carry **4** marks each.)

27. Show that if G is a finite group with even number of elements, then there is an $a \neq e$ in G such that $a^2 = e$.

28. Prove that the set of all 2×2 matrices with entries from R and determinant as '1' is a group under matrix multiplication.

29. Prove that in a group, an element and its inverse have the same order.

30. For every integer $n > 2$, prove that the group $u(n^2 - 1)$ is not cyclic.

31. Show that every permutation on a finite set can be written as a cycle or as a product of cycles.

32. Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$. Write α, β and $\alpha\beta$ as product of disjoint cycles.

33. Prove that for every positive integer n, $Aut(Z_n)$ is isomorphic to $u(n)$.

34. State and prove Fermat's little theorem.

35. Let H be a normal subgroup of a group G and K be any subgroup of G. Then $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G.

36. Let G be a group and $Z(G)$ be the centre of G. Prove that if $G/Z_{(G)}$ is cyclic, then G is abelian.

37. Let $\varphi : G \to \overline{G}$ be a group homomorphism and let $g \in G$. Prove that if $\varphi(g) = g'$, then $\varphi^{-1}(g') = \{x \in G \mid \varphi(x) = g'\} = gKer\varphi$.

38. Find all abelian groups of order 360, upto isomorphism.

SECTION – IV

(Answer any **two** questions. These questions carry **15** marks each)

39. (a) Let * be defined on Q^+ by $a * b = \frac{ab}{4}$. Prove that $(Q,*)$ is an abelian group.

(b) Prove that if a and b are elements of a group G, then the linear equations $ax = b$ and $ya = b$ have unique solutions x and y in G.

40. (a) Show that a nonempty subset H of a group G is a subgroup of G if and only if $ab^{-1} \in H$, for all $a, b \in H$.

(b) Let a be an element of order n in a group and let k be a positive integer. Prove that $\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle$ and $|a^k| = n / \gcd(n,k)$.

41. (a) Prove that the collection of all permutations of a finite set is group under permutation multiplication.

(b) If the pair of cycles $\alpha = (a_1, a_2, ...a_m)$ and $\beta = (b_1, b_2, ...b_n)$ have no entries in common, then $\alpha\beta = \beta\alpha$.

42. Suppose that $\varphi : G \to \overline{G}$ is a group isomorphism. Prove that

(a) For every integer n and for every a in $G, \varphi(a^n) = [\varphi(a)]^n$.

(b) $G = \langle a \rangle$ if and only if $\overline{G} = \langle \varphi(a) \rangle$.

(c) φ carries the identity of G into the identify of $\overline{G}$.

43. (a) State and prove Lagrange's theorem.

(b) Is the converse of Lagrange's theorem true? Justify.

44. Let $\varphi : G \to \overline{G}$ be a group homomorphism and let H be a subgroup of G. Prove that

(a) If H is normal in G, then $\varphi(H)$ is normal in G.

(b) If $|H| = n$, then $|\varphi(H)|$ divides n.

(c) If $\overline{K}$ is a subgroup of $\overline{G}$, then $\varphi^{-1}(\overline{K}) = \{K \in G | \varphi(K) \in \overline{K}\}$ is a subgroup of G.